Polarized Protons in HERA

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Polarized proton beams at HERA can currently only be produced by extracting a beam from a polarized source and then accelerating it in the three synchrotrons at DESY. In this paper, the processes which can depolarize a proton beam in circular accelerators are explained, devices which could avoid this depolarization in the DESY accelerator chain are described, and specific problems which become important at the high energies of HERA are mentioned. At HERA's high energies, spin motion cannot be accurately described with the isolated resonance model which has been successfully used for lower energy rings. To illustrate the principles of more accurate simulations, the invariant spin field is introduced to describe the equilibrium polarization state of a beam and the changes during acceleration. It will be shown how linearized spin motion leads to a computationally quick approximation for the invariant spin field and how to amend this with more time consuming but accurate non-perturbative computations. Analysis with these techniques has allowed us to establish optimal Siberian Snake schemes for HERA.

1. INTRODUCTION

In contrast to polarized high energy electron beams which can become polarized by the emission of spin flip synchrotron radiation, proton beams do not become polarized after acceleration. High energy polarized protons can only be produced by accelerating a beam from a polarized ion source. The highest energy reached so far was 25GeV in the AGS and a lot of care had to be taken during acceleration to preserve this injected polarization [1,3]. To explain depolarization in circular accelerators, some concepts from spin dynamics have to be introduced. When a particle with charge q moves through a magnetic field, the motion of the classical spin vector in the instantaneous rest frame is described by the Lorentz force equation and the Thomas-BMT equation [4,5],

$$\frac{d\vec{p}}{dt} = -\frac{q}{m\gamma}\{\vec{B}_\perp\} \times \vec{p}\,, \qquad \frac{d\vec{s}}{dt} = -\frac{q}{m\gamma}\{(G\gamma+1)\vec{B}_\perp + (G+1)\vec{B}_\parallel\} \times \vec{s}\,, \tag{1}$$

where $\vec{B}_\perp$ and $\vec{B}_\perp$ are the magnetic field components perpendicular and parallel to the particle's momentum $\vec{p}$. At high energy where $(G\gamma+1)/\gamma \approx G$, the spin motion is independent of energy and a fixed field integral of 5.48Tm leads to a spin rotation of 180° , in contrast to the orbit deflection which varies with $1/\gamma$. For fixed orbit deflections and thus fixed ratio of $\vec{B}_\perp/\gamma$, the spin precession rate increases with energy as we now describe.

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In purely transverse magnetic fields, the Thomas-BMT equation has the same structure as the Lorentz force equation up to a factor $G\gamma + 1$. The spin therefore rotates like the momentum but with a magnified rate. At the HERA energy of 920GeV, the magnification factor is $G\gamma = 1756$ so that the spin is rotated by 100° when a proton's direction is altered by 1mrad in a transverse magnetic field and the spin rotates 1756 times around the vertical while a particle makes one turn on the design orbit of a flat circular accelerator, where the fields are vertical. The number of spin rotations which a particle performs while it travels along the closed orbit once is called the spin tune ν_0 .

When the spin tune is integer, the spin comes back to a field imperfection with the same direction after one turn and the effect of the field error can add up coherently from turn to turn. This resonant depolarization at integer spin tunes ν_0 is called an *imperfection resonance* [6].

When viewed at a fixed azimuth θ of the accelerator, the particles appear to perform harmonic oscillations around the closed orbit with the frequencies ν_x, ν_y, and ν_τ for horizontal, vertical, and longitudinal motion. These are called the orbital tunes. Some of the fields through which a particle propagates will therefore oscillate with the orbital tunes. Whenever the non-integer part of the spin tune is equal to plus or minus one of these frequencies, the resulting coherent perturbation can lead to depolarization. The coherent depolarization at the first order resonance condition $\nu_0 = m + \nu_k$ is called an *intrinsic resonance* [6]. Here the notation ν_1=ν_x, ν_2=-ν_x, ν_3=ν_y, ν_4=-ν_y, ν_5=ν_τ, ν_6=-ν_τ is used. Since the spin tune changes with energy (in a flat ring $\nu_0 = G\gamma$) resonances will have to be crossed at some energies during acceleration.

After one turn around the accelerator, all spins of particles on the closed orbit have been rotated by $2\pi\nu_0$ around a unit rotation vector $\vec{n}_0$. This vector is determined by the accelerator's main guide fields and small field imperfections only perturb $\vec{n}_0$ weakly except at energies where the guide fields would produce an integer spin tune. Then, when viewed from a fixed azimuth, spins would come back after one turn apparently without a rotation. Close to imperfection resonances the remaining rotation and therefore the direction of $\vec{n}_0$ will be dominated by the influence of field errors. While the energy changes during acceleration, $\vec{n}_0$ changes its direction strongly at these resonances. Whenever this change is sufficiently slow, spins which are initially parallel to $\vec{n}_0$ will follow the change of $\vec{n}_0$ adiabatically. Imperfection resonances can therefore be crossed either by making the field imperfections small enough or by making them so strong that $\vec{n}_0$ already starts to get influenced by the field errors sufficiently long before the resonance and then changes slowly enough to let all spins follow adiabatically while the resonance is crossed. Special magnets for enhancing this effect without disturbing the orbit are referred to as partial snakes. So far solenoid magnets have been used [7] but for the AGS a helical dipole partial snake is under construction [3].

The motion of spins along phase space trajectories is dominated by the main guide fields on the closed orbit except close to an intrinsic resonance, where the coherent perturbations described above can dominate over the main guide fields. When the emittance of the beam and therefore the amplitude of the perturbations is sufficiently small, intrinsic resonances can be crossed without loss of polarization. Polarization in the core of the beam will therefore be only weakly influenced when crossing intrinsic resonances. If a strong coherent perturbation is slowly switched on and off, an effect similar to adiabatically following $\vec{n}_0$

occurs and polarization is conserved. Therefore, while an intrinsic resonance is crossed, perturbations influencing particles in the tails of a beam will slowly increase already before the resonance and this adiabatic conservation of polarization can occur. In intermediate parts of the beam, however, the polarization is lost. This type of depolarization can be overcome by slowly exciting the whole beam coherently at a frequency close to the orbital tune which causes the perturbation. All spins then follow the adiabatic change of the polarization direction and the resonance can be crossed with little loss of polarization. The excitation amplitude is then reduced slowly so that the beam emittance does not change noticeably during the whole process. This mechanism has recently been tested successfully at the AGS [1]. An older technique of avoiding depolarization at strong intrinsic resonances utilizes pulsed quadrupoles to move the orbital tune within a few microseconds just before a resonance so that the resonance is crossed so quickly that the spin motion is hardly disturbed [8].

So far no polarized beam has been accelerated to more than 25GeV [3]. But the possibility of polarized proton acceleration has been analyzed for RHIC (250GeV), for the TEVATRON (900GeV), and for HERA (920GeV). When accelerating through approximately 5000 resonances in the case of HERA, even very small depolarization in every resonance crossing would add up to a detrimental effect.

It was mentioned below equation (1) that in a fixed transverse magnetic field the deflection angle of high energy particles depends on energy, whereas the spin rotation does not depend on energy. It is therefore possible to devise a fixed field magnetic device which rotates spins by π whenever a high energy particle travels through it at the different energies of an acceleration cycle. Such field arrangements which rotate spins by π while perturbing the orbit only moderately are called Siberian Snakes [9]. The rotation axis is called the snake axis and the angle of this axis to the beam direction is referred to as the snake angle ψ . Let us consider a Siberian Snake with snake angle ψ_1 at one point in a flat ring and a second Siberian Snake with snake angle ψ_2 at the opposite side of the ring where the spin has rotated by $G\gamma/2$. The spin rotation around the vertical between the Siberian Snakes is described with Pauli matrices by the quaternion $\cos(\pi G\gamma/2) + i\sin(\pi G\gamma/2)\sigma_2$. The rotation by the first Siberian Snake is described by $i[\sin(\psi_1)\sigma_1 + \cos(\psi_1)\sigma_3]$. The total rotation for one turn around the ring is then described by

$$\begin{aligned}
& i[\sin(\psi_1)\sigma_1 + \cos(\psi_1)\sigma_3] \cdot [\cos(\pi G\gamma/2) + i\sin(\pi G\gamma/2)\sigma_2] \\
& \cdot i[\sin(\psi_2)\sigma_1 + \cos(\psi_2)\sigma_3] \cdot [\cos(\pi G\gamma/2) + i\sin(\pi G\gamma/2)\sigma_2] \\
= \; & i[\sin(\psi_1 + \pi G\gamma/2)\sigma_1 + \cos(\psi_1 + \pi G\gamma/2)\sigma_3] \\
& \cdot i[\sin(\psi_2 + \pi G\gamma/2)\sigma_1 + \cos(\psi_2 + \pi G\gamma/2)\sigma_3] \\
= \; & -\cos(\psi_1 - \psi_2) + i\sin(\psi_1 - \psi_2)\sigma_2 \; . \qquad (2)
\end{aligned}$$

For $\psi_1 - \psi_2 = \pi/2$ the spins rotate in total 1/2 times around the vertical $\vec{n}_0$ during a complete turn around the ring, giving $\nu_0 = 1/2$. All imperfection resonances and, since the orbital tunes cannot be 1/2, also all first order intrinsic resonances are avoided by the insertion of these two Siberian Snakes, and polarized beam acceleration to very high energy could become possible. Siberian Snakes can only be used at sufficiently high energies since their fields are not changed during acceleration of the beam and they produce orbit distortions which are too big for energies below approximately 8GeV [10].

2. THE DESY ACCELERATOR CHAIN FOR POLARIZED PROTONS

For HERA a polarized proton beam would be produced by a polarized H^- source. Then it would be accelerated to 750keV in an RFQ and then to 50MeV in the LINAC III from where it would be accelerated in the synchrotron DESY III to 7.5GeV/c. In the next ring, PETRA, 40GeV/c are reached, and HERA finally accelerates to 920GeV/c. The four main challenges for obtaining highly polarized beams in HERA are: (1) *Production* of a 20mA pulsed H^- beam. (2) *Polarimetry* at various stages in the acceleration chain. (3) *Acceleration* through the complete accelerator chain with little depolarization. (4) *Storage* of a polarized beam at the top energy over many hours with little depolarization.

Polarized protons are produced either by a polarized atomic beam source (ABS), where a pulsed beam with 87% polarization for 1mA beam current has been achieved, or by an optically pumped polarized ion source (OPPIS), where pulsed beams with 60% for 5mA have been achieved. Experts claim that 80% polarization and 20mA could be achievable with the second type of source. The current source at DESY produces 60mA but the maximal current of 205mA in DESY III can already be achieved with a 20mA source.

Polarimeters will have to be installed at several crucial places in the accelerator chain. The source would contain a Lyman-α polarimeter [11]. Another polarimeter could be installed after the RFQ [12]. This could not be operated continuously since it disturbs the beam. The transfer of polarized particles through the LINAC III could be optimized with a polarimeter similar to that in the AGS LINAC; and like the AGS, DESY III could contain an internal polarimeter [8]. Polarization at DESY III energies has been achieved and measured at several labs already. It is different with PETRA and HERA energies; for these high energies there is no established polarimeter. Here one has to wait and see how the novel techniques envisaged and developed for RHIC will work [13].

Since DESY III has a super period of eight, only 4 strong intrinsic first order resonances have to be crossed. They are at values for the spin tune $G\gamma$ of $8-\nu_y$, $0+\nu_y$, $16-\nu_y$, and $8+\nu_y$. Depolarization can be avoided by jumping the tune with pulsed quadrupoles in a few microseconds or by excitation of a resonance with an RF dipole. A solenoid partial snake would be used to cross the one strong imperfection resonance at $G\gamma = 8$. All these methods have been tested successfully at the AGS and it is likely that a highly polarized proton beam could be extracted from the DESY III synchrotron at 7.5GeV/c .

In PETRA it would be very cumbersome to cross all resonances which can be seen in figure 1 (middle). Since Siberian Snakes can be constructed for the injection energy of PETRA [14] it will be best to avoid all first order resonances with two such devices. There is space for Siberian Snakes in the east and the west section of PETRA.

3. SPECIFIC PROBLEMS FOR THE HERA RING

HERA is a very complex accelerator and a brief look already indicates four reasons why producing a polarized beam in HERA is more difficult than in an ideal ring. (1) HERA has a super periodicity of one and only an approximate mirror symmetry between the North and South halves of the ring. Therefore more resonances appear than in a ring with some higher super periodicity and special schemes for canceling resonances in symmetric lattices [15] cannot be used in such a ring. (2) The proton ring of HERA is on top of the electron ring in the arcs, and the proton beam is bent down to the level of the electron ring on both sides of the three experiments H1, HERMES, and ZEUS in the North, East, and South straight sections. The HERA proton accelerator is therefore not a flat ring. The destructive effect of the vertical bends can, however, be eliminated by so called flattening snakes [16,17] which let the spin motion in pairs of vertical bends cancel and makes $\vec{n}_0$ vertical outside the non-flat sections of HERA. (3) There is space for spin rotators which make the polarization parallel to the beam direction inside the collider experiments while keeping it vertical in the arcs, and there is also space for four Siberian Snakes. But installing more than four Siberian Snakes would involve a lot of costly construction work. Simulations have shown that 8 snakes with properly chosen snake angles would be desirable. However, if one does not choose optimal snake angles, then four-snake-schemes can be better than eight snake schemes [18]. (4) The energy is very high and therefore the spin rotates rapidly. If HERA had been designed for polarized proton acceleration, several parts of the ring would probably have been constructed differently.

4. APPLICABLE THEORY AND SIMULATION TOOLS

4.1. The isolated resonance model

In the isolated resonance model, the field components which perturb the spin of a particle that oscillates around the closed orbit are Fourier expanded. The perturbation of spin motion is then approximated by dropping all except one of the Fourier components. When $\vec{z}$ describes the phase space coordinates relative to the closed orbit and θ describes the accelerator's azimuth, the Thomas-BMT equation (1) has the form $d\vec{s}/d\theta = \vec{\Omega}(\vec{z},\theta)\times\vec{s}$. The precession vector $\vec{\Omega}$ can be written as $\vec{\Omega}_0(\theta) + \vec{\omega}(\vec{z},\theta)$ with a part on the closed orbit and a part which is linear in the phase space coordinates $\vec{z}$. For spins parallel to the rotation vector on the closed orbit $\vec{n}_0(\theta)$ only the components of $\vec{\omega}(\vec{z},\theta)$ which are perpendicular to $\vec{n}_0$ perturb the polarization. We now choose two mutually orthogonal unit vectors $\vec{m}_0$ and $\vec{l}_0$ which are perpendicular to $\vec{n}_0$ and precess around $\vec{\Omega}_0$ according to the Thomas-BMT equation on the closed orbit. The frequency of their rotation is given by the spin tune ν_0.

In this model a depolarizing resonance occures when a Fourier component of $\vec{\omega}(\vec{z}(\theta),\theta)$ rotates with the same frequency as $\vec{m}_0$ and $\vec{l}_0$ so that there is a coherent perturbation of the spins away from $\vec{n}_0$. The Fourier component ϵ_{ν_0} for this frequency is obtained from the Fourier integral along a trajectory $\vec{z}(\theta)$,

$$\epsilon_{\nu_0} = \lim_{N\to\infty}\frac{1}{2\pi N}\int_0^{2\pi N}\vec{\omega}(\vec{z}(\theta),\theta)\cdot(\vec{m}_0 + i\vec{l}_0)d\theta\ . \tag{3}$$

These resonance strengths are shown in the figure 1 (top), (middle), and (bottom) for the three proton synchrotrons at DESY. They were all computed for an oscillation amplitude of $\vec{z}(\theta)$ corresponding to the one sigma vertical emittance of 4πmm mrad.

4.2. The invariant spin field

Already at extraction from PETRA the polarized beam would have somewhat more energy than any other polarized proton beam so far obtained and one has to ask whether the isolated resonance model successfully used so far for describing depolarization is still applicable. To understand whether the isolated resonance model describes spin motion at HERA accurately, we introduce the invariant spin field of a circular accelerator. It has been mentioned that a particle on the closed orbit has to be polarized parallel to $\vec{n}_0$ in order to have the same polarization after every turn. Similarly, one can ask if the whole field of spin directions for particles at different phase space points can be invariant from turn to turn.

Each particle can have a different spin direction at its phase space point $\vec{z}$ and each of these spins propagates with a different precession vector $\vec{\Omega}(\vec{z}(\theta),\theta)$ in the Thomas-BMT equation. A spin field $\vec{n}(\vec{z})$ which is invariant after one turn around the ring is called an invariant spin field or a Derbenev–Kontratenko $\vec{n}$-axis [9]. A beam which is polarized parallel to this invariant spin field at every phase space point does not change its polarization state from turn to turn. Particles change their location in phase space from some initial phase space coordinate $\vec{z}_i$ in the Poincaré section at azimuth θ to some final coordinate after one turn $\vec{z}_f = \vec{M}(\vec{z}_i)$ according to the one turn map. And spins change their directions according to the one turn spin transport matrix $\underline{R}(\vec{z}_i)$, but the invariant field of spin directions $\vec{n}(\vec{z}_i)$ does not change after one turn. This requirement is encompassed by the periodicity condition

$$\vec{n}(\vec{M}(\vec{z}_i)) = \underline{R}(\vec{z}_i)\vec{n}(\vec{z}_i) \ . \tag{4}$$

Note that the polarization state of a particle beam is in general not invariant from turn to turn when all particles are initially completely polarized parallel to each other, but rather when each particle is polarized parallel to $\vec{n}(\vec{z})$ at its phase space point $\vec{z}$. In this case the polarization of a particle will be parallel to $\vec{n}(\vec{z}_i)$ whenever it comes close to its initial phase space point $\vec{z}_i$ during later turns around the ring, as long as $\vec{n}(\vec{z})$ is sufficiently continuous. When two particles travel along the same trajectory, the angle between their two spins does not change. When a particle is initially polarized with an angle ϕ to $\vec{n}(\vec{z})$, it will therefore be rotated around $\vec{n}(\vec{z})$ every time it comes close to $\vec{z}_i$, but it will still have the angle ϕ to the invariant spin field. The time averaged polarization at $\vec{z}_i$ will therefore be parallel to $\vec{n}(\vec{z}_i)$, but it can only have the magnitude 1 if the spin was initially parallel to the invariant spin field. However, even if all particles are initially polarized parallel to $\vec{n}(\vec{z})$, the beam polarization is not 1 but $< \vec{n} >$ where $< \ldots >$ denotes an average over the beam. The maximum average polarization that can be stored in an accelerator at a given fixed energy is therefore $| < \vec{n} > |$. It was first pointed out in [19] that this maximum polarization can be small in HERA.

Since the spin dynamics depends on energy, the invariant spin field $\vec{n}(\vec{z})$ will change during the acceleration process. If this change is slow enough, spins which are parallel to $\vec{n}(\vec{z})$ will follow adiabatically. However, if the change is too rapid, polarization will be lost. It is therefore good to have $< \vec{n} >$ close to 1 not only at the collider energy but during the complete acceleration cycle. Four problems occur when the different directions of $\vec{n}(\vec{z})$ are not close to parallel for all particles in the beam. (1) Sudden changes of $\vec{n}(\vec{z})$ reduces the polarization. (2) The average polarization available to the collider experiment is reduced. (3) The polarization involved in each collision process depends on the phase space position of the interacting particles. (4) Measuring the polarization in the tail of the beam will not give accurate information on the average polarization of the beam.

4.2.1. Linearized spin orbit motion

At azimuth θ, a spin can be described by a usually small complex coordinate α with $\vec{s} = \Re\{\alpha\}\vec{m}_0(\theta) + \Im\{\alpha\}\vec{l}_0(\theta) + \sqrt{1-|\alpha|^2}\vec{n}_0(\theta)$. When the spin coordinates α and the phase space coordinates are linearized, one approximates an initial spin by $\vec{s}_i \approx \Re\{\alpha_i\}\vec{m}_0(0) + \Im\{\alpha_i\}\vec{l}_0(0) + \vec{n}_0(0)$ at azimuth 0 and the final spin after one turn around the accelerator by $\vec{s}_f = \Re\{\alpha_f\}\vec{m}_0(0) + \Im\{\alpha_f\}\vec{l}_0(0) + \vec{n}_0(0)$ where α_f is determined by the 7×7 one turn transport matrix $\underline{M}_{77}$,

$$\begin{pmatrix}\vec{z}_f\\ \alpha_f\end{pmatrix} = \underline{M}_{77}\begin{pmatrix}\vec{z}_i\\ \alpha_i\end{pmatrix} = \begin{pmatrix}\underline{M} & \vec{0}\\ \vec{G}^T & e^{i2\pi\nu_0}\end{pmatrix}\begin{pmatrix}\vec{z}_i\\ \alpha_i\end{pmatrix} , \tag{5}$$

whereby $\underline{M}$ is the 6×6 dimensional one turn transport matrix for the phase space variables, the exponential describes the rotation of the spin components α by the spin tune ν_0 around $\vec{n}_0$, the row vector $\vec{G}^T$ describes the dependence of spin motion on phase space motion to first order, and the 6 dimensional zero vector $\vec{0}$ shows that the effect of Stern Gerlach forces on the orbit motion is not considered.

We now write the components perpendicular to $\vec{n}_0$ of the invariant spin field as a complex function $n_\alpha(\vec{z})$ and use a 7 dimensional vector $\vec{n}_1$ to obtain the first order expansion of $\vec{n}(\vec{z})$. The linearized periodicity condition for the invariant spin field is

$$\vec{n}_1(\vec{z}) = \begin{pmatrix}\vec{z}\\ n_\alpha(\vec{z})\end{pmatrix} , \quad \vec{n}_1(\underline{M}\vec{z}) = \underline{M}_{77}\vec{n}_1(\vec{z}) . \tag{6}$$

This equation can be solved for $\vec{n}_1$ after the matrices are diagonalized. Let $\underline{A}^{-1}$ be the column matrix of eigenvectors of the one turn matrix $\underline{M}$. The diagonalized matrix of orbit motion $\underline{\Lambda} = \underline{A}\ \underline{M}\ \underline{A}^{-1}$ has the diagonal elements $\exp(i2\pi\nu_k)$ given by the orbital tunes $\nu_1 = \nu_x$, $\nu_2 = -\nu_x$, etc. We now need the 7×6 dimensional matrix $\underline{T}$ which is the column matrix of the first 6 eigenvectors of $\underline{M}_{77}$ and has the form

$$\underline{T} = \begin{pmatrix}\underline{A}^{-1}\\ \vec{B}^T\end{pmatrix} , \quad \underline{T}\,\underline{\Lambda} = \underline{M}_{77}\underline{T} , \tag{7}$$

where the 7th components of the eigenvectors form a vector $\vec{B}$. If a linear function $\vec{n}_1(\vec{z}) = \underline{K}\vec{z}$ of the phase space coordinates can be found, which satisfies the periodicity condition (6), then an invariant spin field has been determined. Inserting the form $\vec{z}_1 = \underline{K}\vec{z}$ into equation (6) and multiplying the resulting condition $\underline{K}\ \underline{M} = \underline{M}_{77}\underline{K}$ by $\underline{A}^{-1}$ from the right leads to $\underline{K}\ \underline{A}^{-1}\underline{\Lambda} = \underline{M}_{77}\underline{K}\underline{A}^{-1}$. Therefore $\underline{K}\ \underline{A}^{-1}$ is the 7×6 dimensional matrix of eigenvectors $\underline{T}$ satisfying equation (7) and we conclude that there exists a unique linear invariant spin field given by

$$\vec{n}_1(\vec{z}) = \underline{T}\ \underline{A}\vec{z} . \tag{8}$$

In the linear approximation of spin motion, the invariant spin field is simply computed via the eigenvectors of the 7×7 spin orbit transport matrix. This matrix $\underline{M}_{77}$ can be computed in various ways, for example by multiplying the individual spin transport matrixes of all elements [20] or by concatenating spin transport quaternions of individual elements as done in the program SPRINT [21]. In the normal form space belonging to the diagonal matrix $\underline{\Lambda}$ the coordinates are given by the actions J_j and the angle variables Φ_j with

$$\underline{A}\vec{z} = \begin{pmatrix} \sqrt{J_1}e^{i\Phi_1} \\ \sqrt{J_1}e^{-i\Phi_1} \\ \cdots \end{pmatrix} . \tag{9}$$

The average over all angle variables of a phase space torus then leads to the average opening angle of

$$< \phi(\vec{n}, \vec{n}_0) > \approx \mathrm{atan}(\sqrt{< |n_\alpha|^2 >}) = \mathrm{atan}(\sqrt{\sum_{k=1}^{3}(|B_{2k-1}|^2 + |B_{2k}|^2)J_k}) , \tag{10}$$

where the B_k are the 7th components of the eigenvectors in equation (7).

These opening angles are shown for DESY III in figure 2 (top) and it is apparent that at the places where resonant spin perturbations are described by a large resonance strength, the invariant spin field has a large opening angle. It is obvious when comparing with the resonance strength of figure 1 (top) that the influence of different resonances does not overlap in the linearized spin approximation. At PETRA energies of up to 40GeV, the resonances already come very close to each other as seen when comparing figure 2 (middle) with figure 1 (middle) and one can only barely expect an isolated resonance approximation to lead to accurate results. For high energies between 780 and 820GeV/c in HERA, figure 2 (bottom) clearly shows that one cannot speak of isolated resonances. Often the influences of 4 resonances overlap.

The approximation of linearized spin motion contains all first order orbital frequencies, since it linearized the precession vector $\vec{\Omega}$ with respect to $\vec{z}$. However, in contrast to the isolated resonance model, none of these resonances is ignored and the effect of overlapping resonances can be seen.

It is possible to recover the first order isolated resonance strength from the one turn spin orbit transport matrix. In analogy to the complex notation for the spin component perpendicular to $\vec{n}_0$, the perturbing precession vector $\vec{\omega}$ is expressed by a complex function $\omega(\vec{z}, \vec{\theta})$ as $\vec{\omega} = \Re\{\omega\}\vec{m}_0 + \Im\{\omega\}\vec{l}_0 + (\vec{\omega}\cdot\vec{n}_0)\vec{n}_0$. Inserting this into the Thomas-BMT equation, one obtains

$$\alpha' = -i\sqrt{1-\vec{\alpha}^2}\omega + i\alpha(\vec{\omega}\cdot\vec{n}_0) . \tag{11}$$

In the case of spins which are nearly parallel to $\vec{n}_0$, one can linearize in α and $\vec{z}$. For a spin which was initially parallel to $\vec{n}_0$ one obtains $\alpha(\theta) \approx -i\int_0^\theta \omega d\theta$. Comparing with equation (3) one can express the resonance strength as $\epsilon_{\nu_0} = i\lim_{N\to\infty}\frac{1}{2\pi N}\alpha(2\pi N)$. The resonance strength can therefore be computed from $\underline{M}_{77}^N/N$ for large N. The computation becomes very efficient if one uses $\underline{M}_{77}^{2N} = (M_{77}^N)^2$ iteratively.

The coordinate vectors $\vec{m}_0(2\pi)$ and $\vec{l}_0(2\pi)$ to which $\alpha(2\pi)$ refers have rotated by the spin tune ν_0, whereas the final spin coordinate α_f computed by $\underline{M}_{77}$ refers to the coordinate vectors $\vec{m}_0(0)$ and $\vec{l}_0(0)$. Therefore $\alpha(2\pi N) = \alpha_f \exp(-i2\pi N\nu_0)$, and ϵ_{ν_0} can be computed from powers of the one turn matrix, which can most efficiently be evaluated in diagonal form,

$$\epsilon_{\nu_0} \;=\; i\lim_{N\to\infty}\frac{1}{2\pi N}\alpha(2\pi N) = i\lim_{N\to\infty}\frac{1}{2\pi N}(0, e^{-iN2\pi\nu_0})\begin{pmatrix} \underline{M} & 0 \\ \vec{G}^T & e^{i2\pi\nu_0} \end{pmatrix}^N \begin{pmatrix} \vec{z} \\ 0 \end{pmatrix} \tag{12}$$

$$= \quad i \lim_{N\to\infty} e^{-iN2\pi\nu_0} \frac{1}{2\pi N} \sum_{j=0}^{N-1} [e^{i(N-j-1)2\pi\nu_0} \vec{G}^T \underline{A}^{-1} \underline{\Lambda}^j] \underline{A}\vec{z} \tag{13}$$

$$= \quad i e^{-i2\pi\nu_0} G_l A_{lk}^{-1} A_{km} z_m \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{j=0}^{N-1} e^{i2\pi j(\nu_k - \nu_0)} \tag{14}$$

where one has to sum over equal indices k, l, and m. This formula shows that the resonance strength is always zero, except at a resonance condition $\nu_0 = m + \nu_k$. At such a spin tune, the resonance strength is given by

$$2\pi|\epsilon_{\nu_0=\nu_k}| = |\vec{G}^T \underline{A}^{-1} \underline{\mathrm{diag}}(0...1...0) \underline{A}\vec{z}| = |\vec{G}^T \underline{A}^{-1} (0...\sqrt{J_k} e^{i\Phi_k}...0)^T| = |\vec{G}\cdot\vec{v}_k| \sqrt{J_k} \ . \tag{15}$$

The 1 in the diagonal matrix is in position k. Here $\underline{A}^{-1}(0...\sqrt{J_k}e^{i\Phi_k}...0)^T$ is the initial value for a phase space trajectory which has only Fourier components with frequencies ν_k plus integers and the kth eigenvector $\vec{v}_k$ of $\underline{M}$ has been used. The infinite Fourier integral in equation (3) has been reduced to the scalar product between the bottom vector of $\underline{M}_{77}$ and an eigenvector of $\underline{M}$. This very simple formula is used in the program SPRINT.

4.3. Non-perturbative methods

While one does not drop Fourier coefficients in the approximation of linearized spin motion, there are other limitations. The approximation is no longer justified when $|n_\alpha|$ becomes large, which happens close to resonances in the figures 2. Therefore the validity of linearized spin motion had to be be checked by computing the invariant spin field non-perturbatively. In the last few years two iterative higher order and three non-perturbative methods of computing the invariant spin field have been developed [22]. All of these methods agree within their ranges of mutual applicability. The invariant spin field obtained from a non-perturbative method contains the effect of all Fourier coefficients in $\vec{\Omega}$. When comparing this spin field with $\vec{n}_1$, it was found that linearized orbit motion describes the opening angle and thus the maximum storeable polarization well in domains where the opening angle is small. At the critical energies, where the maximum polarization is low during the acceleration process, non-perturbative methods become essential for simulation and results obtained with the computationally quick linearization of spin motion should always be checked with more time consuming non-perturbative methods if possible.

One application of this strategy is the filtering method [23]. Four or eight Siberian Snakes are inserted into HERA to fix the spin tune to 1/2 for all energies and to let $\vec{n}_0$ be vertical in the flat arcs. These conditions do not fix all snake angles. Currently there is, however, no established formula to determine good snake angles. Since the opening angle of $\vec{n}(\vec{z})$ is such a critical quantity for high energy polarized proton acceleration, we have decided to maximize $< \vec{n} >$ by choosing snake angles. A computer code was written which tested approximately 10^6 snake schemes and computational speed was therefore essential. Linearized spin motion was used to find the 8-snake-schemes with smallest average value of $|n_\alpha|$ over the acceleration cycle. These filtered snake schemes then also had relatively small opening angles when computed non-perturbatively with stroboscopic averaging [21].

Two other indications showed that this filtering leads to good snake schemes. (1) Tracking simulations of the complete ramp process showed that the snake schemes found by filtering leads to less depolarization [24] than other schemes which were initially proposed. (2) Computation of the amplitude dependent spin tune, which can only be performed when

$\vec{n}(\vec{z})$ has been found non-perturbatively, shows that snakes schemes found by filtering have significantly less spin tune spread over orbital amplitudes than other proposed schemes [25]. With the optimal scheme for four Siberian Snakes in HERA it turned out to be possible to accelerate in computer simulations approximately 65% of the beam to high energy with little loss of polarization as long as no closed orbit distortions were present [18]. In simulations, the current 1mm rms closed orbit distortions lead to depolarization [26]. Therefore either the closed orbit will have to be controlled more accurately or techniques which make the spin motion less sensitive to closed orbit distortion [27] will have to be utilized.

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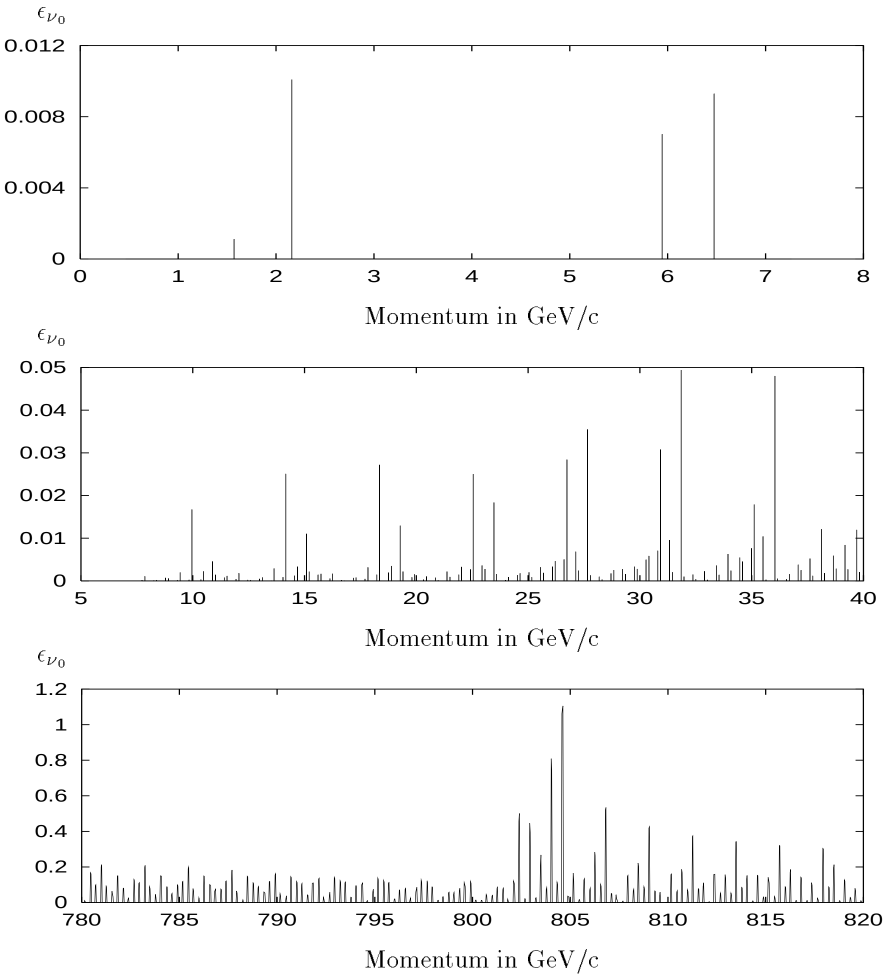

Figure 1: Resonance strength for DESY III, PETRA, and HERA.

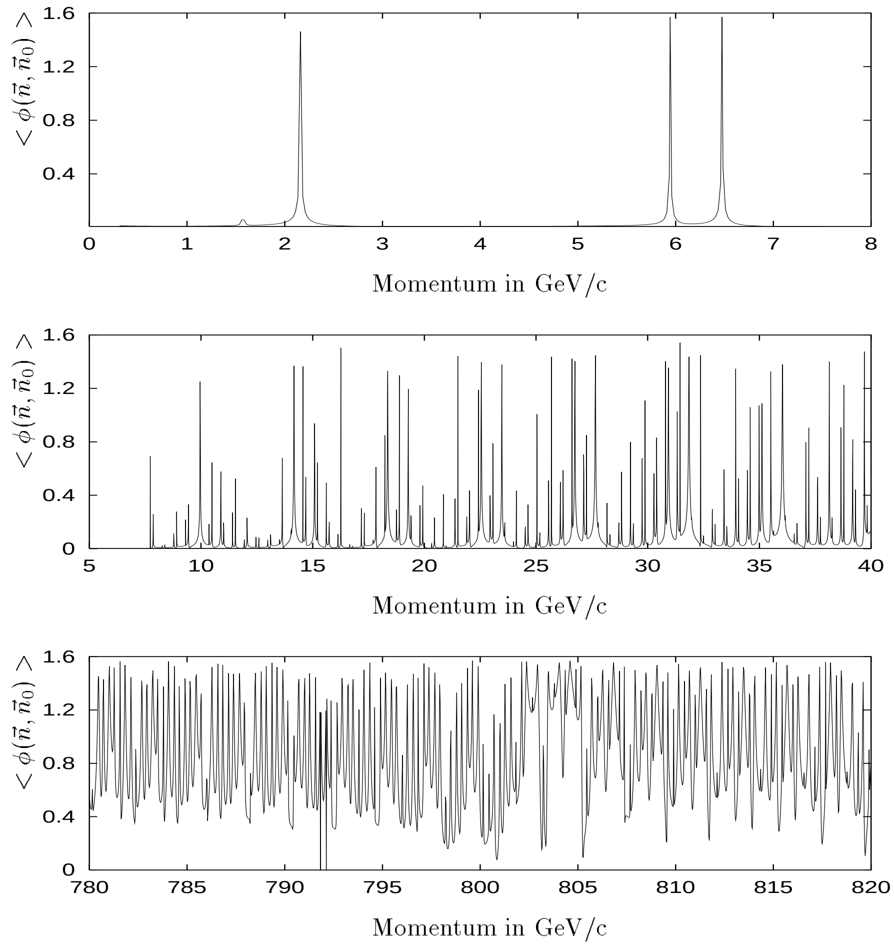

Figure 2: Opening angle for DESY III, PETRA, and HERA.